New similarity solutions for the modified Boussinesq equation



1989 J. Phys. A: Math. Gen. 22 2355

(http://iopscience.iop.org/0305-4470/22/13/029)

J. Phys. A: Math. Gen. **22** (1989) 2355–2367. Printed in the UK

# New similarity solutions for the modified Boussinesq equation

Peter A Clarkson
Department of Mathematics, University of Exeter, Exeter, EX4 4QE, UK

Received 28 November 1988

**Abstract.** In this paper we present some new similarity solutions of the modified Boussinesq (mBq) equation, which is a completely integrable soliton equation. These new similarity solutions include reductions to the second and fourth Painlevé equations which are not obtainable using the standard Lie group method for finding group-invariant solutions of partial differential equations; they are determined using a new and direct method which involves no group theoretical techniques.

## 1. Introduction

In this paper we discuss similarity solutions of the modified Boussinesq (mBq) equation

$$q_{tt} - q_t q_{xx} - \tfrac{1}{2} q_x^2 q_{xx} + q_{xxxx} = 0. \tag{1.1}$$

The mBq equation is a soliton equation solvable by inverse scattering (Quispel *et al* 1982) and there is a 'Miura type' transformation relating solutions of (1.1) to solutions of the Boussinesq equation (Boussinesq 1871, 1872)

$$u_{tt} + a u_{xx} + b(u^2)_{xx} + c u_{xxxx} = 0 \tag{1.2}$$

where $a$, $b$, $c$ are constants, which arises in several physical applications and also is a soliton equation solvable by inverse scattering (Zakharov 1974, Ablowitz and Haberman 1975, Caudrey 1980, 1982, Deift *et al* 1982). Specifically, if $q(x,t)$ satisfies the mBq equation (1.1), then using the Bäcklund transformation (Hirota and Satsuma 1977)

$$v_x(x,t) = -q_t + \sqrt{3} q_{xx} - \tfrac{1}{2} q_x^2 \tag{1.3a}$$

$$v_t(x,t) = \sqrt{3} q_{xt} + q_{xxx} - q_x q_t - \tfrac{1}{6} q_x^3 + \delta \tag{1.3b}$$

where $\delta$ is a constant, it is easily shown that $v(x,t)$ is a solution of the potential Boussinesq equation

$$v_{tt} + v_x v_{xx} + v_{xxxx} = 0 \tag{1.4}$$

and $u(x,t) = v_x(x,t)$ is a solution of the Boussinesq equation (1.2) with $a = 0$, $b = 1/2$, $c = 1$, which we can assume without loss of generality (see also Quispel *et al* 1982 and Gromak 1987).

  

The inverse scattering method, in effect, reduces the solution of the non-linear partial differential equation to that of a linear integral equation, and the partial differential equation is then said to be *completely integrable.* Furthermore, completely integrable partial differential equations all seem to possess several remarkable properties including elastically interacting soliton solutions, the existence of infinitely many independent conservation laws and symmetries, Bäcklund transformations, Lax representation, the Painlevé property, etc; however, the precise relationship betwen these properties has yet to be rigorously established (cf Ablowitz and Segur (1981) and Calogero and Degasperis (1982); see Hirota and Satsuma (1977), Quispel *et al* (1982), Clarkson (1986) and Gromak (1987) for the derivation of some of these properties for the mBq equation).

The classical method of finding similarity solutions of a given partial differential equation is to use the Lie group method of infinitesimal transformations (sometimes called the method of group-invariant solutions), originally due to Lie (1891) (for recent descriptions of this method see Bluman and Cole (1974), Olver (1986), Ovsiannikov, (1982), Winternitz (1983)). Although the method is entirely algorithmic, it often involves a large amount of tedious algebra and auxilliary calculations which can become virtually unmanageable if attempted manually, and so recently symbolic manipulation programs have been developed, both in MACSYMA (Rosenau and Schwarzmeier 1979, Champagne and Winternitz 1985) and in REDUCE (Schwarz 1985), in order to facilitate the determination of the associated similarity solutions. (See Schwarz (1988) for a review of the use of computer algebra to find symmetries of differential equations.)

Bluman and Cole (1969) proposed a generalisation of Lie's method which they called the 'non-classical method of group-invariant solutions', which itself has recently been generalised by Olver and Rosenau (1986, 1987). All these three methods determine Lie point symmetries of a given partial differential equation since the infinitesimals depend only on the independent and dependent variables.

One common characteristic of all these methods for finding symmetries and associated similarity solutions of a given partial differential equation is the use of group theory.

In this paper we use a direct method of deriving similarity solutions of partial differential equations which has recently been developed by Clarkson and Kruskal (1989). The unusual characteristic about this method in comparison with the others mentioned above is that it involves no use of group theory. This method has been successfully applied to obtain new similarity solutions of the Boussinesq equation (1.2) (Clarkson and Kruskal 1989) and in this paper we use it to derive new similarity solutions of the mBq equation (1.1). Essentially the basic idea is to seek a solution of a given partial differential equation in the form

$$q(x,t) = Q(x,t,w(z(x,t))) \tag{1.5}$$

which is the most general form for a similarity solution. Then we require that substitution of (1.5) into the partial differential equation yield an ordinary differential equation for $w(z)$. This imposes conditions upon $Q$ and its derivatives which enable one to solve for $Q$. It turns out that it is sufficient for the mBq equation (1.1) to seek similarity solutions in the special form

$$q(x,t) = \alpha(x,t) + \beta(x,t)w(z(x,t)). \tag{1.6}$$

The outline of this paper is as follows: in § 2 we describe the classical similarity solutions of the mBq equation; in § 3 we derive new similarity solutions; and in § 4 we discuss the results.

## 2. Classical similarity solutions

Firstly we shall derive the classical similarity solutions of the mBq equation (1.1) using the Lie group method as given by Bluman and Cole (1974). Consider the one-parameter ($\varepsilon$) Lie group of infinitesimal transformations in $(x, t, q)$ given by

$$\xi = x + \varepsilon X(x,t,q) + O(\varepsilon^2) \tag{2.1a}$$

$$\tau = t + \varepsilon T(x,t,q) + O(\varepsilon^2) \tag{2.1b}$$

$$\eta = q + \varepsilon Q(x,t,q) + O(\varepsilon^2) \tag{2.1c}$$

$$\eta_\xi = q_x + \varepsilon Q^x + O(\varepsilon^2) \tag{2.2a}$$

$$\eta_\tau = q_t + \varepsilon Q^t + O(\varepsilon^2) \tag{2.2b}$$

$$\eta_{\xi\xi} = q_{xx} + \varepsilon Q^{xx} + O(\varepsilon^2) \tag{2.2c}$$

$$\eta_{\xi\xi\xi\xi} = q_{xxxx} + \varepsilon Q^{xxxx} + O(\varepsilon^2) \tag{2.2d}$$

where the functions $Q^x$, $Q^t$, $Q^{xx}$, $Q^{tt}$, $Q^{xxxx}$ are determined from equations (2.1) (cf Bluman and Cole 1974). The mBq equation (1.1) is invariant under this transformation if $\eta(\xi, \tau)$ satisfies the same equation as $q(x, t)$. Substituting (2.1) and (2.2) into the mBq equation for $\eta(\xi, \tau)$, then to first order in $\varepsilon$ we have

$$Q^{tt} - Q^t q_{xx} - q_t Q^{xx} - q_x Q^x q_{xx} - \tfrac{1}{2} q_x^2 Q^{xx} + Q^{xxxx} = 0. \tag{2.3}$$

The infinitesimals $X(x,t,q)$, $T(x,t,q)$, $Q(x,t,q)$ are determined by collecting coefficients of like derivative terms in $q$ and equating them to zero in equations (2.3), and the following infinitesimals are obtained:

$$X = \alpha x + \beta \qquad T = 2\alpha t + \gamma \qquad Q = \delta \tag{2.4}$$

(cf Schwarz 1988), where $\alpha$, $\beta$, $\gamma$, $\delta$ are arbitrary constants. Similarity solutions are obtained by solving the characteristic equations

$$\frac{\mathrm{d}x}{X(x,t,q)} = \frac{\mathrm{d}t}{T(x,t,q)} = \frac{\mathrm{d}u}{Q(x,t,q)}. \tag{2.5}$$

For the mBq equation (1.1), there are two cases to consider.

(i) $\alpha = 0$. Integrating equations (2.5) yields the travelling wave solution

$$q(x,t) = \int^z f(z_1)\,\mathrm{d}z_1 \qquad z = \gamma x - \beta t \tag{2.6}$$

where $f(z)$ satisfies

$$\gamma^4 f'' + \beta^2 f + \tfrac{1}{2}\gamma^2 \beta f^2 - \tfrac{1}{6}\gamma^4 f^3 = A \tag{2.7}$$

with $' := \mathrm{d}/\mathrm{d}z$ and $A$ an arbitrary constant of integration. If $\gamma \neq 0$, then $f(z)$ is solvable in terms of elliptic functions, whilst if $\gamma = 0$, then (2.7) is a simple linear equation.

(ii) $\alpha \neq 0$. Integrating equations (2.5) yields the scaling solution

$$q(x,t) = \frac{\delta}{2\alpha}\ln\left(t+\frac{\gamma}{2\alpha}\right) + \int^{z} g(z_1)\,\mathrm{d}z_1 \qquad z = \frac{x+\beta/\alpha}{(t+\gamma/2\alpha)^{1/2}}$$

where $g(z)$ satisfies

$$g''' - \frac{1}{2}g^2g' + \frac{z}{2}gg' - \frac{\delta}{2\alpha}g' + \frac{z^2}{4}g' + \frac{3z}{4}g - \frac{\delta}{2\alpha} = 0. \tag{2.8}$$

If we make the transformation

$$g(z) = -3^{3/4}Y(X) - z \qquad X = 3^{1/4}z/2 \tag{2.9}$$

then $Y(X)$ satisfies the fourth Painlevé equation (cf Ince 1956)

$$\frac{\mathrm{d}^2Y}{\mathrm{d}X^2} = \frac{1}{2Y}\left(\frac{\mathrm{d}Y}{\mathrm{d}X}\right)^2 + \tfrac{3}{2}Y^3 + 4XY^2 + 2(X^2-A)Y + \frac{B}{Y} \tag{2.10}$$

where $A = -4\delta/9\sqrt{3}\alpha$ and $B$ is a constant of integration (cf Quispel *et al* 1982, Gromak 1987).

However, the mBq equation (1.1) also possesses similarity solutions which are *not* obtained using the classical Lie group method. For example, as shown by Quispel *et al* (1982), the mBq equation (1.1) also possesses the similarity solution

$$q(x,t) = x(\lambda+\mu t) + \int^{z} h(z_1)\,\mathrm{d}z_1 \qquad z = x - \lambda t - \tfrac{1}{2}\mu t^2 \tag{2.11}$$

where $\lambda, \mu$ are constants and $h(z)$ satisfies

$$h'' - \tfrac{1}{6}h^3 - (\mu z - \tfrac{1}{2}\lambda^2)h = a \tag{2.12}$$

where $a$ is a constant of integration (though Quispel *et al* (1982) did not point out that this similarity solution is not obtainable using the classical Lie group method). Equation (2.12) is equivalent (through translation and scaling of the variables) to the second Painlevé equation (cf Ince 1956)

$$\frac{\mathrm{d}^2Y}{\mathrm{d}X^2} = 2Y^3 + XY + A \tag{2.13}$$

where $A$ is an arbitrary constant, unless $\mu = 0$, in which case it is solvable in terms of elliptic functions. The infinitesimals which give rise to the similarity solution (2.11) are

$$X(x,t,q) = \lambda + \mu t \qquad T(x,t,q) = 1 \qquad Q(x,t,q) = (\lambda+\mu t)^2 + \mu x \tag{2.14}$$

which are clearly *not* a special case of the infinitesimals obtained using the classical Lie group method (2.4), unless $\mu = 0$. Since this describes a Lie point transformation of the mBq equation, then it is probably obtainable using the 'non-classical method' due to Bluman and Cole (1969). The non-classical method involves more algebra and auxiliary calculations than the classical Lie method; in fact, as suggested by Olver and Rosenau (1987), the determining equations for these 'non-classical' symmeteries for some partial differential equations might be too difficult to solve explicitly. The principal reason for this is that the determining equations for $X$, $T$, $Q$ are a linear system of equations in the classical case whereas they are a non-linear system in the non-classical case. Furthermore, for some equations such as the linear heat equation, it is well known that the non-classical method does not yield any more similarity reductions than the classical method does (Bluman and Cole 1969, see also Ames 1972, Hill 1982).

## 3. New similarity solutions

In this section we seek solutions of the mBq equation (1.1) in the form

$$q(x,t) = \alpha(x,t) + \beta(x,t)w(z(x,t)) \tag{3.1}$$

where $\alpha(x,t)$, $\beta(x,t)$ and $z(x,t)$ are assumed to be sufficiently differentiable functions and $w(z)$ is four-times differentiable. (We shall show below why it is sufficient to seek a similarity solution of the mBq equation (1.1) in the form (3.1) rather than the more general form $q(x,t) = Q(x,t,w(z(x,t)))$.)

Substituting (3.1) into (1.1) and collecting coefficients of like derivatives and powers of $w(z)$ yields

$$\begin{aligned}
\beta z_x^4 w'''' &+ (6\beta z_x^2 z_{xx} + 4\beta_x z_x^3)w''' \\
&+ \left[\beta(3z_{xx}^2 + 4z_x z_{xxx}) + 12\beta_x z_x z_{xx} + 6\beta_{xx} z_x^2 - \tfrac{1}{2}\alpha_x \beta z_x^2 - \alpha_t \beta z_x^2 + \beta z_t^2\right] w'' \\
&+ \left[\beta z_{xxxx} + 4\beta_x z_{xxx} + 6\beta_{xx} z_{xx} + 4\beta_{xxx} z_x - \tfrac{1}{2}\alpha_x^2(2\beta_x z_x + \beta z_{xx}) - \alpha_x \alpha_{xx} \beta z_x \right.\\
&\left. - \alpha_t(2\beta_x z_x + \beta z_{xx}) - \alpha_{xx}\beta z_t + \beta_t z_t + \beta z_{tt}\right] w' \\
&+ \left[\beta_{xxxx} - \alpha_x \alpha_{xx} \beta_x - \tfrac{1}{2}\alpha_x^2 \beta_{xx} - \alpha_t \beta_{xx} - \alpha_{xx}\beta_t + \beta_{tt}\right] w \\
&- \tfrac{1}{2}\left[\beta z_x^2 w'' + (2\beta_x z_x + \beta z_{xx})w' + \beta_{xx} w + \alpha_{xx}\right] \\
&\times \left[\beta^2 z_x^2 (w')^2 + 2\beta\beta_x z_x w w' + \beta_x^2 w^2\right] \\
&+ \left[\beta z_x^2 w'' + (2\beta_x z_x + \beta z_{xx})w' + \beta_{xx} w\right](\beta_t w + \beta z_t w') + \alpha_{tt} - \alpha_t \alpha_{xx} - \tfrac{1}{2}\alpha_x^2 \alpha_{xx} + \alpha_{xxxx} \\
&= 0
\end{aligned} \tag{3.2}$$

where $' := \mathrm{d}/\mathrm{d}z$. In order that this equation is an ordinary differential equation for $w(z)$ then the ratios of different derivatives and powers of $w(z)$ have to be functions of $z$ only. This gives an overdetermined system of equations for $\alpha(x,t)$, $\beta(x,t)$, $z(x,t)$, whose solutions yield the desired similarity solutions. Before doing this we make some remarks about this direct method.

*Remark 3.1.* We shall use the coefficient of $w''''$ (i.e. $\beta z_x^4$) as the normalising coefficient and require that the other coefficients are of the form $\beta z_x^4 \Gamma(z)$, where $\Gamma$ is a function of $z$ to be determined.

*Remark 3.2.* Whenever we use an upper case Greek letter to denote a function (e.g. $\Gamma(z)$), then this is a function to be determined upon which we can perform any mathematical function (e.g. differentiation, integration, take logarithm, exponentiate, take power, rescale, etc) and then also call the resulting function $\Gamma(z)$ without loss of generality (e.g. the differential of $\Gamma(z)$ will be called $\Gamma(z)$).

*Remark 3.3.* There are three freedoms in the determination of $\alpha$, $\beta$, $z$ which we can exploit, without loss of generality:

(*a*) if $\alpha(x,t)$ is of the form $\alpha = \alpha_0(x,t) + \beta(x,t)\Gamma(z)$, where $\alpha_0$ is specified and $\Gamma(z)$ is any function, then we can assume that $\Gamma \equiv 0$ (make the transformation $w(z) \to w(z) - \Gamma(z)$);

(*b*) if $\beta(x,t)$ is of the form $\beta = \beta_0(x,t)\Gamma(z)$, where $\beta_0$ is specified and $\Gamma(z)$ is any function, then we can assume that $\Gamma \equiv 1$ (make the transformation $w(z) \to w(z)/\Gamma(z)$);

(*c*) if $z(x,t)$ is defined by an equation of the form $\Gamma(z) = z_0(x,t)$, where $z_0$ is specified and $\Gamma(z)$ is any invertible function, then we can also assume that $\Gamma \equiv z$ (make the transformation $z \to \Gamma^{-1}(z)$, where $\Gamma^{-1}$ is the inverse of $\Gamma$).

We shall now proceed to determine the general similarity solutions of the mBq equation using this method.

The coefficient of $(w')^2 w''$ yields the constraint

$$\beta z_x^4 \Gamma(z) = \beta^3 z_x^4$$

where $\Gamma(z)$ is a function to be determined. Hence, using the freedom mentioned in remark 3.3(*b*) above,

$$\beta \equiv 1. \tag{3.3}$$

The coefficient of $w'''$ yields the constraint

$$\beta z_x^4 \Gamma(z) = 4\beta_x z_x^3 + 6\beta z_x^2 z_{xx}$$

where $\Gamma(z)$ is to be determined. Hence using (3.3) and rescaling $\Gamma$, we have

$$z_x \Gamma(z) + z_{xx}/z_x = 0$$

which upon integration gives

$$z_x \Gamma(z) = \Theta(t) \tag{3.4}$$

where $\Theta(t)$ is a function of integration (recall remark 3.2). Integrating (3.4) gives

$$\Gamma(z) = x\Theta(t) + \Sigma(t)$$

where $\Sigma(t)$ is another function of integration. Using the freedom mentioned in remark 3.3(*c*), we have

$$z = x\theta(t) + \sigma(t) \tag{3.5}$$

where $\theta(t)$ and $\sigma(t)$ are to be determined.

The coefficient of $w'w''$ yields the constraint

$$\beta z_x^4 \Gamma(z) = \beta^2 z_x^2 (z_t + \alpha_x z_x)$$

where $\Gamma(z)$ is to be determined, and using equations (3.3) and (3.5), this simplifies to

$$\theta\Gamma(z) = \frac{1}{2\theta}\left(x\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}\sigma}{\mathrm{d}t}\right) + \alpha_x.$$

Integrating this and using the freedom mentioned in remark 3.3(*a*), we have

$$\alpha = -\frac{1}{2\theta}\left(x^2\frac{\mathrm{d}\theta}{\mathrm{d}t} + 2x\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right) + \phi(t) \tag{3.6}$$

where $\phi(t)$ is to be determined.

If $\alpha$, $\beta$ and $z$ are as given in equations (3.6), (3.3) and (3.5), respectively, then equation (3.2) simplifies to

$$\theta^4(w'''' - \tfrac{1}{2}(w')^2w'') + \frac{1}{2}\left[x^2\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 2x\theta\frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2} + \left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 - 2\theta^2\frac{\mathrm{d}\phi}{\mathrm{d}t}\right]w'' + \frac{1}{2}\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}(w')^2$$
$$+\left(x\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2}\right)w' + \alpha_{tt} - \alpha_t\alpha_{xx} - \tfrac{1}{2}\alpha_x^2\alpha_{xx} + \alpha_{xxxx} = 0. \tag{3.7}$$

This is an ordinary differential equation for $w(z)$ provided that

$$\theta^3\gamma_1(z) = \frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{3.8a}$$
$$\theta^4\gamma_2(z) = x\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2} \tag{3.8b}$$
$$\theta^4\gamma_3(z) = x^2\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 2x\theta\frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2} + \left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 - 2\theta^2\frac{\mathrm{d}\phi}{\mathrm{d}t} \tag{3.8c}$$
$$\theta^4\gamma_4(z) = \alpha_{tt} - \alpha_t\alpha_{xx} - \tfrac{1}{2}\alpha_x^2\alpha_{xx} \tag{3.8d}$$

where $\gamma_1(z)$, $\gamma_2(z)$, $\gamma_3(z)$, $\gamma_4(z)$ are to be determined. Firstly consider equation (3.8*a*), since $z = x\theta(t) + \sigma(t)$, then necessarily $\gamma_1(z) = A$, constant, and hence $\theta(t)$ satisfies

$$\mathrm{d}\theta/\mathrm{d}t = A\theta^3. \tag{3.9}$$

Using this, it is easily shown from equations (3.6) and (3.8) that

$$\alpha(x,t) = -\frac{x^2A\theta^2}{2} - \frac{x}{\theta}\frac{\mathrm{d}\sigma}{\mathrm{d}t} + \phi \tag{3.10a}$$
$$\gamma_2(z) = 3A^2z + B \tag{3.10b}$$
$$\gamma_3(z) = 3A^2z^2 + 2Bz + C \tag{3.10c}$$
$$\gamma_4(z) = -(\tfrac{9}{2}A^3z^2 + 3ABz + \tfrac{3}{2}AC) \tag{3.10d}$$

where $B$, $C$ are constants, and $\sigma(t)$ and $\phi(t)$ satisfy

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2} = (3A^2\sigma + B)\theta^4 \tag{3.11}$$
$$\left(\frac{d\sigma}{dt}\right)^2 - 2\theta^2\frac{\mathrm{d}\phi}{\mathrm{d}t} = \theta^4(3A^2\sigma^2 + 2B\sigma + C). \tag{3.12}$$

Therefore the general similarity solution of the mBq equaions (1.1) is given by

$$q(x,t) = w(z) - \tfrac{1}{2}Ax^2\theta^2 - \frac{x}{\theta}\frac{\mathrm{d}\sigma}{\mathrm{d}t} + \phi \qquad z(x,t) = x\theta(t) + \sigma(t) \tag{3.13}$$

where $\theta(t)$, $\sigma(t)$ and $\phi(t)$ satisfy equations (3.9), (3.11) and (3.12), respectively, and $W(z) := w'(z)$ satisfies

$$W''' - \tfrac{1}{2}W^2W' + \tfrac{1}{2}AW^2 + (3A^2z+B)W + \tfrac{1}{2}(3A^2z^2+2Bz+C)W' - \left(\tfrac{9}{2}A^3z^2 + 3ABz + \tfrac{3}{2}AC\right) = 0. \tag{3.14}$$

There are two cases to consider.

(i) $A = 0$. In this case, the solutions of equations (3.9), (3.11) and (3.12) are

$$\theta(t) \equiv 1 \tag{3.15a}$$
$$\sigma(t) = \tfrac{1}{2}Bt^2 + \lambda_1 t + \lambda_0 \tag{3.15b}$$
$$\phi(t) = \tfrac{1}{2}(\lambda_1^2 - 2B\lambda_0 - C)t + \lambda_2 \tag{3.15c}$$

where $\lambda_0$, $\lambda_1$, $\lambda_2$ are arbitrary constants and the associated similarity solution of the mBq equation is

$$q(x,t) = w(z) - x(Bt + \lambda_1) + \tfrac{1}{2}(\lambda_1^2 - 2B\lambda_0 - C)t + \lambda_2 \tag{3.16a}$$
$$z = x + \tfrac{1}{2}Bt^2 + \lambda_1 t + \lambda_0 \tag{3.16b}$$

where $W(z) := w'(z)$ satisfies

$$W'' - \tfrac{1}{6}W^3 + (Bz + \tfrac{1}{2}C)W = D \tag{3.17}$$

where $D$ is a constant of integration. Dependent on whether $B = 0$ or not, the solution of this equation is expressible either in terms of elliptic functions or the second Painlevé equation (cf Ince 1956)

$$\mathrm{d}^2Y/\mathrm{d}X^2 = 2Y^3 + XY + A$$

where $A$ is a constant. Without loss of generality, we can set $\lambda_0 = 0$ and $\lambda_2 = 0$ in equations (3.15) and then this similarity solution reduces to one given by Quispel *et al* (1982), which we discussed in § 2 above.

(ii) $A \neq 0$. In this case, we set $A = -1/2$ and $B = 0$ without loss of generality, therefore solving equations (3.9) and (3.11) yields

$$\theta(t) = (t + t_0)^{-1/2} \tag{3.18a}$$
$$\sigma(t) = \lambda_1(t + t_0)^{3/2} + \lambda_2(t + t_0)^{-1/2} \tag{3.18b}$$

where $t_0$, $\lambda_1$, $\lambda_2$ are arbitrary constants (without loss of generality assume that $t_0 = 0$ and $\lambda_2 = 0$). Hence solving equation (3.12), we obtain the similarity solution

$$q(x,t) = w(z) + \frac{x^2}{4t} - \frac{3\lambda_1 xt}{2} + \frac{\lambda_1^2 t^3}{4} - \frac{C}{2}\ln t \tag{3.19a}$$
$$z = xt^{-1/2} + \lambda_1 t^{3/2} \tag{3.19b}$$

where $W(z) := w'(z)$ satisfies

$$W''' - \tfrac{1}{2}W^2W' - \tfrac{1}{4}W^2 + \tfrac{3}{4}zW + (\tfrac{3}{8}z^2 + \tfrac{1}{2}C)W' + \tfrac{9}{16}z^2 + \tfrac{3}{4}C = 0. \tag{3.20}$$

Now make the transformation

$$W(z) = -3^{3/4}Y(X) - z \qquad X = 3^{1/4}z/2 \tag{3.21}$$

then $Y(X)$ satisfies the fourth Painlevé equation (cf Ince 1956)

$$\frac{\mathrm{d}^2Y}{\mathrm{d}X^2} = \frac{1}{2Y}\left(\frac{\mathrm{d}Y}{\mathrm{d}X}\right)^2 + \frac{3}{2}Y^3 + 4XY^2 + 2(X^2 - A)Y + \frac{B}{Y} \tag{3.22}$$

where $A = -4C/9\sqrt{3}$ and $B$ is a constant of integration (cf Quispel *et al* 1982 and Gromak 1987).

The similarity solution (3.19) represents a new reduction of the mBq equation to the fourth Painlevé equation. The infinitesimals associated with this similarity solution are

$$X(x,t,q) = x - 3\lambda_1 t^2 \tag{3.23a}$$
$$T(x,t,q) = 2t \tag{3.23b}$$
$$Q(x,t,q) = 6\lambda_1^2 t^3 - 6\lambda_1 xt - C \tag{3.23c}$$

which (like the infinitesimals for the similarity solutions obtained in (i) above, equation (2.14)) are clearly *not* a special case of the infinitesimals obtained using the classical Lie group method (2.4), unless $\lambda_1 = 0$.

Finally in this section, we shall show why it is sufficient to seek a similarity solution of the mBq equation (1.1) in the special form (3.1) rather than the more general form

$$q(x,t) = Q(x,t,w(z(x,t))). \tag{3.24}$$

Suppose we seek a similarity solution in the form (3.24), then substituting (3.24) into (1.1) yields

$$\begin{aligned}
&\left[Q_{tt} + 2Q_{tw}w'z_t + Q_{ww}(w')^2z_t^2 + Q_w(w''z_t + w'z_{tt})\right] \\
&\qquad - \left\{(Q_t + Q_w w'z_t) + \tfrac{1}{2}\left[Q_x^2 + 2Q_xQ_w w'z_x + Q_w^2(w')^2z_x^2\right]\right\} \\
&\qquad \times \left[Q_{xx} + 2Q_{xw}w'z_x + Q_{ww}(w')^2z_x^2 + Q_w(w''z_x + w'z_{xx})\right] \\
&\qquad + Q_{xxxx} + 4Q_{xxxw}w'z_x + 6Q_{xxww}(w')^2z_x^2 + 4Q_{xwww}(w')^3z_x^3 + Q_{wwww}(w')^4z_x^4 \\
&\qquad + 6Q_{xxw}(w'z_{xx} + w''z_x^2) + 12Q_{xww}\left[w'w''z_x^3 + (w')^2z_xz_{xx}\right] \\
&\qquad + 6Q_{www}\left[(w')^2w''z_x^4 + (w')^3z_x^2z_{xx}\right] + 4Q_{xw}(w'''z_x^3 + 3w''z_xz_{xx} + w'z_{xxx}) \\
&\qquad + Q_{ww}\left\{\left[4w'w''' + 3(w'')^2\right]z_x^4 + 18w'w''z_x^2z_{xx} + (w')^2(4z_xz_{xxx} + 3z_{xx}^2)\right\} \\
&\qquad + Q_w\left[w''''z_x^4 + 6w'''z_x^2z_{xx} + w''(4z_xz_{xxx} + 3z_{xx}^2) + w'z_{xxxx}\right] \\
&= 0
\end{aligned}$$

where $' := \mathrm{d}/\mathrm{d}z$. In order that this is an ordinary differential equation in $w(z)$, then the ratios of different derivatives of $w(z)$ have to be functions of $w$ and $z$. If we use the coefficient of $w''''$ (i.e. $Q_wz_x^4$) as the normalising coefficient, then the coefficients of $w'w'''$ and $(w'')^2$ require that

$$Q_wz_x^4\Gamma(w,z) = Q_{ww}z_x^4$$

where $\Gamma(w,z)$ is a function to be determined. Hence

$$\Gamma(w,z) = Q_{ww}/Q_w$$

which after integrating twice yields

$$Q(x,t,w) = \Theta(x,t)\Gamma(w,z) + \Phi(x,t)$$

where $\Theta(x,t)$, $\Phi(x,t)$ are arbitrary functions. Therefore it is sufficient to seek similarity solutions of the mBq equation (1.1) in the form (3.1).

## 4. Discussion

Firstly we make the following general remarks about similarity solutions of partial differential equations.

*Remark 4.1.* Generally, given a partial differential equation with a symmetry (i.e. a transformation of the dependent and/or independent variables that leaves the equation invariant), the action of the symmetry group takes a solution of the equation into another solution of the equation. Starting with a fixed solution that corresponds to the identity element of the group, every element of the group corresponds to some solution of the equation. The starting solution can be *any* solution of the equation. This mapping can be used to *define* a symmetry, and the group carries the set of all solutions of the partial differential equation into itself.

*Remark 4.2.* Given such a symmetry of a partial differential equation, one can seek solutions which are mapped into themselves under the action of the group. These are *similarity solutions* corresponding to the group. For a partial differential equation with two independent and one dependent variables, these solutions typically are solutions of an ordinary differential equation.

*Remark 4.3.* Alternatively, the ordinary differential equation can be taken as a means of generating special solutions of the partial differential equation, without regard to what maps into what. Then the ordinary differential equation appears to be an example of the *side condition* introduced by Olver and Rosenau (1986, 1987). This seems to be the way similarity solutions are generally used.

*Remark 4.4.* The special (or similarity) solutions obtained in this paper are also defined through an ordinary differential equation that is 'compatible' with the partial differential equation (in the sense that they have common solutions). Again, the ordinary differential equation is a side condition on the partial differential equation, and the surprise is that there are common solutions. The issue of mapping solutions of the partial differential equation does *not* arise in the procedure and so there is no connection with remarks 4.1 and 4.2 above (in fact, as shown below, the additional special/similarity solutions do not map solutions of the partial differential equation into solutions of the same partial differential equation). The method appears to be any effective way of producing special/similarity solutions.

*Remark 4.5.* The special/similarity solutions obtained in this paper are associated with Lie point transformations and it remains an open question as to whether these new special/similarity solutions and their associated symmetries can be obtained using any of the other generalisations of the classical Lie method; such as the 'non-classical' method due to Bluman and Cole (1969) (or its generalisation due to Olver and Rosenau 1987), and the method developed recently by Bluman *et al* (1988). However, even if these special/similarity solutions are theoretically obtainable by any of these methods (and it is not immediately obvious that this is necessarily the case), then it seems that this direct method of finding special/similarity solutions is somewhat simpler to implement; in fact, it appears to be simpler even than the classical Lie point method (without the assistance of a symbolic manipulation program).

It is easily shown that any similarity solution of the mBq equation obtained in this paper which is not obtained by the classical Lie group method does *not* have the property that the associated group maps solutions of the mBq equation into itself.

Firstly, consider the similarity solution

$$q(x,t) = w(z) + x(\lambda_1 + \mu_1 t) \qquad z = x - \lambda_1 t - \tfrac{1}{2}\mu_1 t^2. \tag{4.1}$$

The associated one-parameter ($\epsilon$) group is given by

$$x \to x + (\lambda_1 + \mu_1 t)\epsilon + \tfrac{1}{2}\mu_1\epsilon^2 \tag{4.2a}$$

$$t \to t + \epsilon \tag{4.2b}$$

$$q \to q + [x\mu_1 + (\lambda_1 + \mu_1 t)^2]\epsilon + \tfrac{3}{2}\mu_1(\lambda_1 + \mu_1 t)\epsilon^2 + \tfrac{1}{2}\mu_1^2\epsilon^3. \tag{4.2c}$$

This group maps solutions of the mBq equation (1.1) into solutions of

$$q_{tt} - q_t q_{xx} - \tfrac{1}{2}q_x^2 q_{xx} + q_{xxxx} + 2\mu_1\epsilon[q_{xt} + (\lambda_1 + \mu_1 t)q_{xx} - \mu_1] = 0. \tag{4.3a}$$

Note that if $q(x,t)$ is as defined in (4.1) (i.e. it is the similarity solution), then

$$q_{xt} + (\lambda_1 + \mu_1 t)q_{xx} - \mu_1 \equiv 0. \tag{4.3b}$$

Therefore, unless $\mu_1 \equiv 0$, then the group (4.2) does not map solutions of the mBq equation into itself, though it does give similarity solutions.

An analogous result holds for the similarity solution

$$q(x,t) = v(z) - 2\lambda_2 xt - \mu_2 \ln t \qquad z = xt^{-1/2} + \lambda_2 t^{3/2} \tag{4.4}$$

(this is equivalent to (3.19)). The associated one-parameter ($\gamma$) group is given by

$$x \to xe^{\gamma} + \lambda_2 te^{\gamma}(1 - e^{3\gamma}) \tag{4.5a}$$

$$t \to te^{2\gamma} \tag{4.5b}$$

$$q \to q + 2\lambda_2 xt(1 - e^{3\gamma}) - 2\lambda_2^2 t^3 e^{3\gamma}(1 - e^{3\gamma}) - 2\mu_2\gamma. \tag{4.5c}$$

This group maps solutions of the mBq equation (1.1) into solutions of

$$q_{tt} - q_t q_{xx} - \tfrac{1}{2}q_x^2 q_{xx} + q_{xxxx} = \lambda_2(1 - e^{-3\gamma})[4tq_{xt} + 2q_x + (2x - 6\lambda_2 t^2)q_{xx} + 12\lambda_2 t]. \tag{4.6}$$

Additionally, if $q(x,t)$ is as defined in equation (4.4) (i.e. it is the similarity solution), then

$$4tq_{xt} + 2q_x + (2x - 6\lambda_2 t^2)q_{xx} + 12\lambda_2 t \equiv 0.$$

Hence, unless $\lambda_2 \equiv 0$, then the group (4.5) does not map solutions of the mBq equation into itself, though it does give similarity solutions.

For both these similarity solutions, the associated group maps solutions of the mBq equation (1.1) into solutions of a 'perturbed mBq equation', in which the associated similarity solution identically satisfies the perturbed part of the equation. This poses the question: what type of symmetries of the mBq equation (1.1) are those which we have obtained (and are not found using the classical Lie method)?

This direct method of determining similarity solutions of given partial differential equations poses another open question: what is the relationship (if any) between the direct method used in this paper and the other generalisations of the classical Lie method, such as those due to Bluman and Cole (1969), Olver and Rosenau (1986, 1987) and Bluman *et al* (1988)? In their generalisation of the non-classical method of Bluman and Cole (1969), Olver and Rosenau (1986, 1987) showed that in order to determine a group-invariant solution of a given partial differential equation, one could use *any* group of infinitesimal transformations whatsoever. Generally, given any group of infinitesimal transformations and any partial differential equation, there will be *no* solutions of the partial differential equation which are invariant under the group and so the question becomes how does one determine *a priori* whether a given group will give a meaningful similarity reduction? One possibility is that by seeking a solution of a given partial differential equation in a certain form (as we have done in this paper), one is naturally led to the appropriate group (i.e. the requirement that the similiarity reduction reduces the partial differential equation to an ordinary differential equation is equivalent to the *side condition* in the terminology of Olver and Rosenau (1986, 1987)). The results obtained both here and in an earlier paper (Clarkson and Kruskal 1989) support the conclusions drawn by Olver and Rosenau (1986) that 'the unifying theme behind finding special solutions of partial differential equations is not, as is commonly suppposed, group theory, but rather the more analytic subject of overdetermined systems of partial differential equations'. However, group theory clearly remains important in the determination of explicit, physically significant, special solutions of partial differential equations (as demonstrated by Olver and Rosenau (1987)).

It appears that the mathematically and physically relevant determination of special/similarity solutions of partial differential equations will continue to attract considerable research interest.

## Acknowledgments

I would like to thank Martin Kruskal, Peter Olver, Harvey Segur and Pavel Winternitz for helpful discussions.

## References

Ablowitz M J and Haberman R 1975 *J. Math. Phys.* **16** 2301–5

Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia, PA: SIAM)

Ames W F 1972 *Nonlinear Partial Differential Equations in Engineering* vol II (New York: Academic)

Bluman G W and Cole J D 1969 *J. Math. Mech.* **18** 1025–42

——1974 *Similarity Methods for Differential Equations* (*Applied Mathematical Science Series* **13**) (Berlin: Springer)

Bluman G W, Kumei S and Reid G J 1988 *J. Math. Phys.* **29** 806–11

Boussinesq J 1871 *Comptes Rendus* **72** 755–9

——1872 *J. Math. Pures Appl.* **7** 55–108

Calogero F and Degasperis A 1982 *Spectral Transforms and Solitons. I* (Amsterdam: North-Holland)

Caudrey P J 1980 *Phys. Lett.* **79A** 264–8

——1982 *Physica* **6D** 51–66

Champagne B and Winternitz P 1985 *Preprint Montreal CRM-1278*

Clarkson P A 1986 *Physica* **19D** 447–50

Clarkson P A and Kruskal M D 1989 *J. Math. Phys.* submitted

Deift P, Tomei C and Trubowitz E 1982 *Commun. Pure Appl. Math.* **35** 567–628

Gromak V I 1987 *Diff. Eqn.* **23** 506–13

Hill J M 1982 *Solution of Ordinary Differential Equations by means of One-Parameter Groups* (*Research Notes in Mathematics* **63**) (London: Pitman)

Hirota R and Satsuma J 1977 *Prog. Theor. Phys.* **57** 797–807

Ince E L 1956 *Ordinary Differential Equations* (New York: Dover)

Lie S 1891 *Vorlesungen über Differentialgleichungen mit Bekannten Infinitesimalen Transformationen* (Leipzig: Teunber) (Reprinted 1967 New York: Chelsea)

Olver P J 1986 *Applications of Lie Groups to Differential Equations* (*Graduate Texts in Mathematics* **107**) (Berlin: Springer)

Olver P J and Rosenau P 1986 *Phys. Lett.* **114A** 107–12

——1987 *SIAM J. Appl. Math.* **47** 263–78

Ovsiannikov L V 1982 *Group Analysis of Differential Equations* (New York: Academic)

Quispel G R W, Nijhoff F W and Capel H W 1982 *Phys. Lett.* **91A** 143–5

Rosenau P and Schwarzmeier JL 1979 *Courant Institute Report* COO-3077-160, MF-94

——1986 *Phys. Lett.* **115A** 75–7

Schwarz F 1985 *Computing* **34** 91–106

——1988 *SIAM Rev.* **30** 450–81

Winternitz P 1983 *Nonlinear Phenomena* (*Lecture Notes in Physics* **189**) ed K B Wolf (Berlin: Springer) pp 263–331

Zakharov V E 1974 *Sov. Phys.-JETP* **38** 108–10